Theory of games

Introduction:

Life is full of struggle and competitions. A great variety of competitive situations is commonly seen in everyday life. For example, candidates fighting an election have their conflicting interests, because each candidate is interested to secure more votes than those secured by all others. Besides such pleasurable activities in competitive situations, we come across much more earnest competitive situations, of military battles, advertising and marketing campaigns by competing business firms, etc. what shoul d be the bid to win a big Government contract in the pace of competition from several contractors? Game must be thought of, in a broad sense, not as a kind of sport but as competitive situation. A kind of conflict in which somebody must win and somebody must lose.

Game theory is a type of decision theory in which one's choice of action is determined after taking into account all possible alternatives available to an opponent playing the same game rather than just by the possibilities of several outcomes.

The mathematical analysis of competitive problems is fundamentally based upon the '**minimax (maximin)** criterion' of **J. Von Neumann** (called the father of game theory). This criterion implies the assumption of rationality from which it is argued that each player will act so as to '**maximize his minimum gain**' or '**minimize his maximum loss'**. The difficulty lies in the deduction from the assumption of 'rationality' that the other player will maximize his minimum gain. There is no agreement even among game theo rists that rational players should so act. In fact, rational players do not act apparently in this way or in any consistent way. Therefore the game theory is generally interpreted as an "**as if**" theory, that is, as if rational decision maker (player) behaved in some well defined (but arbitrarily selected) way, such as *maximize the minimum gain.*

The game theory has only been capable of analyzing very simple competitive situations. Thus, there has been a great gap between what the theory can handle and most actual competitive situations in industry and elsewhere. So the primary contribution of game theory has been its concepts rather than its formal application to solving real problems.

Definition 1: *Game is defined as an activity between two or more persons involving activities by each person according to a set of rules, at the end of which each person receives some benefit or satisfaction or suffers loss (negative profit).*

Definition 2: *The set of rules defines the game. Going through the set of rules once by the participants defines a play.*

A competitive situation will be called a 'Game', if it has the following *properties*:

I. There are a finite number of competitors (participants) called players.
II. Each player has a finite number of strategies (alternatives) available to him.
III. A play of the game takes place when each player employs his strategy.
IV. Every game results in an outcome, e.g., loss or gain or a draw, usually called payoff, to some player.

Basic Definitions:

1. Player: The competitors in the game are known as players. A player may be individual or group of individuals, or an organisation.

2. Strategy: A strategy of a player has been loosely defined as a rule for decision-making in advance of all the plays by which he decides the activities he should adopt. In other words, a strategy for a given player is a set of rule (programmes) that specifies which of the available course of action he should make at each play. This strategy may be of two kinds:

(i) ***Pure strategy***: *It is a decision, in advance of all plays, always to choose a particular course of action*. If a player knows exactly what the other player is going to do, a deterministic situation is obtained and objective function is to maximize the gain. The players select the same strategy each time. Therefore, the pure strategy is a decision rule always to select a particular course of action. A pure strategy is usually represented by a number with which the course of action is associated.

(ii) ***Mixed strategy***: *It is a decision, in advance of all plays, to choose a course of action for each play in accordance with some particular probability distribution*. If a player is guessing as to which activity is to be selected by the other on any particular occasion, a probabilistic situation is obtained and objective function is to maximize the expected gain. Thus, mixed strategy is a selection among pure strategies with fixed probabilities.

3.Optimum strategy: A course of action or play which puts the player in the most preferred position, irrespective of the strategy of his competitors, is called an optimum strategy. **(or)** If the payoff matrix v_{ij} has the saddle point (r,s) then player A and B are said to have r^{th} and s^{th} optimal strategies respectively. **(or)** The course of action which maximizes the profit of a player or minimizes his loss is called an optimal strategy.

4. Zero-sum and Non-zero-sum games: Competitive games are classified according to the number of players involved, i.e., as a two person game, three person game, etc. Another important distinction is between zero-sum and non-zero-sum games. If the players make payments only to each other, i.e., the loss of one is the gain of others, and nothing comes from outside, the competitive game is said to be zero-sum.

Mathematically, suppose an n-person game is played by n players P_1, P_2, ...,P_n whose respective pay-offs at the end of a play of the game are $v_1, v_2, ..., v_n$ then, the game will be called zero-sum if $\sum vi = 0$ at each of the game.

A game which is not zero-sum is called a non-zero-sum game. Most of the competitive games are zero-sum games. An example of a non-zero-sum game is the 'poker' game in which a certain part of the pot is removed from the 'house' before the final payoff.

5. Two-person, Zero-sum (or rectangular) games: A game with only two players (say, Player A and Player B) is called a 'two-person, zero-sum game' if the losses of one player are equivalent gains of the other, so that the sum of their net gains is zero. Two-person, zero-sum games are also called rectangular games as these are usually represented by a payoff matrix in rectangular form.

6. Value of the game: It is the expected payoff of play when all the players of the game follow their optimum strategies. *The game is called fair if the value of the game is zero and unfair, if it is not zero*.**(or)** The payoff (v_{rs}) at the saddle point (r,s) is called the value of the game and is equal to maximin value ($\underline{v}$) and minimax value ($\bar{v}$) of the game.

7. Payoff matrix: Suppose the player A has m activities and the player B has n activities. Then a payoff matrix can be formed by adopting the following rules:

I. Row designations for each matrix are activities available to player A.
II. Column designations for each matrix are activities available for player B.
III. Cell entry 'v_{ij}' is the payment to player A in A's payoff matrix when A chooses the activity i and B chooses the activity j.
IV. With a 'zero-sum, two-person game', the cell entry in the player B's payoff matrix will be negative of the corresponding cell entry 'v_{ij}' in the player A's payoff matrix so that the sum of payoff matrices for player A and player B is ultimately zero.

Table 1: The player A's payoff matrix

Player B

		1	2	. .	j	. .	N
	1	v_{11}	v_{12}	.	v_{1j}	. .	v_{1n}
	2	v_{21}	v_{22}	.	v_{2j}	. .	v_{2n}
	.	.	.	.		. .	
Player A	.	.	.	.		. .	
	i	v_{i1}	v_{i2}	.	v_{ij}	. .	v_{in}
	.	.	.	.		. .	
	.	.	.	.		. .	
	m	v_{m1}	v_{m2}	.	v_{mj}	. .	v_{mn}

Table 2: The player B's payoff matrix

Player B

		1	2	. .	j	. .	N
	1	$-v_{11}$	$-v_{12}$	.	$-v_{1j}$	. .	$-v_{1n}$
	2	$-v_{21}$	$-v_{22}$	.	$-v_{2j}$	. .	$-v_{2n}$
	.	.	.	.		. .	
Player A	.	.	.	.		. .	
	i	$-v_{i1}$	$-v_{i2}$	.	$-v_{ij}$	. .	$-v_{in}$
	.	.	.	.		. .	
	.	.	.	.		. .	
	m	$-v_{m1}$	$-v_{m2}$	.	$-v_{mj}$	. .	$-v_{mn}$

Further, there is no need to write the B's payoff matrix as it is just the negative of A's payoff matrix in a zero-sum two-person game. Thus, if 'v_{ij}' is the gain to A, then '$-v_{ij}$' will be the gain to B.

Example:

In order to make the above concepts clear, consider the coin matching game involving two players only. Each player selects either a head H or a tail T. If the outcomes match (H, H or T, T), A wins Re 1 from B, otherwise B wins Re 1 from A. This game is a two-person zero-sum game, since the winning of one player is taken as the losses for the other. Each has his choices between two pure strategies (H or T). This yields the following (2×2) payoff matrix to player A.

A's payoff matrix is given by:

Player B

		H	T
Player A	H	+1	-1
	T	-1	+1

Characteristics of Game theory:

There can be various types of games that can be classified on the basis of the following characteristics:

I. **Chance of strategy:** If in a game, activities are determined by skill, it is said to be a game of strategy; if they are determined by chance, it is a game if chance. In general, a game may involve game of strategy as well as a game of chance.
II. **Number of persons:** A game is called an n-person game if the number of persons playing is n. The person means an individual or a group aiming at a particular objective.
III. **Number of activities:** These may be finite or infinite.
IV. **Number of alternatives (choices) available to each person** in a particular activity may also be finite or infinite. A finite game has a finite number of alternatives, otherwise the game is said to be infinite.
V. **Information to the players about the past activities of other players** is completely available, partly available, or not available at all.
VI. **Payoff:** The outcome of playing a game is called payoff **(or)** A quantitative measure of satisfaction a person gets at the end of each play is called a payoff. It is a real-valued function of variables in the game. Let v_i be the payoff to the player P_i, $1 \le i \le n$, in an n-person game. If $\sum vi = 0$, then the game is said to be a zero-sum game.

Saddle point: A saddle point is an element of the payoff matrix, which is both the smallest element in its row and the largest element in its column. Furthermore, the saddle point is also regarded as an equilibrium point in the theory of games. **(or)** *A saddle point of a payoff matrix is the position of such an element in the payoff matrix which is minimum in its row and maximum in its column*.

Mathematically, if a payoff matrix (v_{ij}) is such that $\underbrace{min}_{j}[\underbrace{max}_{i}\{v_{ij}\}] = \underbrace{max}_{i}[\underbrace{min}_{j}\{v_{ij}\}]$ then the matrix is said to have a saddle point (r,s) = v_{rs} say.

NOTE:

- ✓ A game is said to be strictly determinable if maximin value ($\underline{v}$) = minimax value ($\bar{v}$) = v (in minimax – maximin principle).
- ✓ A game is said to be fair, if maximin value ($\underline{v}$) = minimax value ($\bar{v}$) = 0.

<u>Rules for determining a saddle point:</u>

Step 1: Select the minimum element of each row of the payoff matrix and put them in $\square$

Step 2: Select the greatest element of each column of the payoff matrix and put them in $\bigcirc$

Step 3: If there appears an element in the payoff matrix marked by '$\square$' and '$\bigcirc$' both, the position of that element is a 'saddle point' of the payoff matrix.

MAXIMIN-MINIMAX PRINCIPLE :(for the selection of the optimal strategies by the two players).

The simplest type of game is one where the best strategies for both players are **pure strategies**. This is the case if and only if, the payoff matrix contains a saddle point.

For player A, minimum value in each row represents the least gain (payoff) to him if he chooses his particular strategy. These are written in the matrix by row minima. He will then select the strategy that maximizes his minimum gains. This choice of player A is called the maximin principle, and the corresponding gain is called the maximin value of the game.

For player B, on the other hand, likes to minimize his losses. The maximum value in each column represents the maximum loss to him if he chooses his particular strategy. These are written in the matrix by column minima. He will then select the strategy that minimizes his maximum losses. This choice of player B is called the minimax principle, and the corresponding loss is the minimax value of the game.

If the maximin value equals the minimax value, then the game is said to have a saddle (equilibrium) point and the corresponding strategies are called optimum strategies. The amount of payoff at an equilibrium point is known as the value of the game.

Theorem 1.1: Let(v_{ij}) be the m×n payoff matrix for a two-person zero-sum game. If $\underline{v}$ denotes the maximin value and $\bar{v}$ denotes the minimax value of the game, then $\bar{v} \geq \underline{v}$. That is,

$$\underbrace{min}_{j}[\underbrace{max}_{i}\{v_{ij}\}] \geq \underbrace{max}_{i}[\underbrace{min}_{j}\{v_{ij}\}].$$

Proof: We have

$$\underbrace{max}_{i}\{v_{ij}\} \geq v_{ij} \qquad for\ all\ j = 1, 2, \dots, n$$

and $$\underbrace{min}_{j}\{v_{ij}\} \leq v_{ij} \qquad for\ all\ i = 1, 2, \dots, m$$

Let the above maximum and minimum values be attained at $i = i_1$ and $j = j_1$, respectively, i.e.,

$$\underbrace{max}_{i}\{v_{ij}\} = v_{i_1 j} \quad and \quad \underbrace{min}_{j}\{v_{ij}\} = v_{ij_1}$$

Then we must have

$$v_{i_1 j} \geq v_{ij} \geq v_{ij_1} \quad for\ all\ j = 1, 2, \dots, n; i = 1, 2, \dots, m.$$

From this, we get

$$\underbrace{min}_{j} v_{i_1 j} \geq v_{ij} \geq \underbrace{max}_{i} v_{ij_1} \quad for\ all\ j = 1, 2, \dots, n; i = 1, 2, \dots, m.$$

Therefore,

$$\underbrace{min}_{j}[\underbrace{max}_{i}\{v_{ij}\}] \geq \underbrace{max}_{i}[\underbrace{min}_{j}\{v_{ij}\}].$$

Example 1.1: Consider a two-person zero-sum game matrix which represents payoff to the player A. Find the optimal strategy, if any.

Player B

		I	II	III	IV	V
Player A	**I**	-2	0	0	5	3
	II	4	2	1	3	2
	III	-4	-3	0	-2	6
	IV	5	3	-4	2	-6

Table: payoff matrix for example 1.1

Solution:

Player B

		I	II	III	IV	V	Row minimum	
	I	$\boxed{-2}$	0	0	5	3	$\boxed{-2}$	
Player A	**II**	4	2	$\boxed{1}$	3	2	$\boxed{1}$	←Maximin
	III	$\boxed{-4}$	-3	0	-2	6	$\boxed{-4}$	
	IV	5	3	-4	2	$\boxed{-6}$	$\boxed{-6}$	
	Column maximum	5	3	1	5	6		
				↑Minimax				

Table: *Player A's payoff matrix*

We use the maximin-minimax principle to determine the optimal strategy. The player A wishes to obtain the largest possible v_{ij} by choosing one of his activities (I, II, III, IV), while the player B is determined to make A's gain the minimum possible by choice of activities from his list (I, II, III, IV, V). The player A is called the *maximizing player* and B, the *minimizing player*. If player A chooses the activity I then it could happen that player B also chooses his activity I. In this case, the player B can guarantee a gain of at least -2 to player A, i.e., *min* {-2, 0, 0, 5, 3} = -2. Similarly, for other choices of player A, i.e., activities II, III and IV, B can force the player A to gain only 1, -4 and -6, respectively, by proper choices from (II, III, IV) i.e., *min* {4, 2, 1, 3, 2} = 1, *min* {-4 , -3, 0, -2, 6} = -4 and *min* { 5, 3, -4, 2, -6} = -6. For player A, minimum value in each row represents the least gain to him if he chooses his particular strategy. These are written in table by row minimum. Player A will select the strategy that maximizes his minimum gains, i.e., *max* {-2, 1, -4, -6} = 1 i.e., player A chooses the strategy II. This choice of player A is called the maximin principle, and the corresponding gain (here 1) is called the maximin value of the game. In general, the player A should try to maximize his least gains or to find $\underbrace{max}_{i}\underbrace{min}_{j}\{ v_{ij}\} = \underline{v}$.

For player B, on the other hand, likes to minimize his losses. The maximum value in each column represents the maximum loss to him if he chooses his particular strategy. These are written in above table by column maximum. Player B will then select the strategy that minimizes his maximum losses. This choice of player B is called the minimax principle, and the corresponding loss is the minimax value of the game. In this case, the value is also 1 and player B chooses the strategy III. In general, the player B should try to minimize his maximum loss or to find $\underbrace{min}_{j}\underbrace{max}_{i}\{v_{ij}\} = \bar{v}$.

If the maximin value is equal to minimax value then the game is said to have a saddle point (here (II, III) cell) and the corresponding strategies are called optimum strategies. The amount at the saddle point is known as the value of the game.

Example 1.2: Solve the game whose payoff matrix is given below:

Player B

		I	II	III
	I	-2	15	-2
Player A	II	-5	-6	-4
	III	-5	20	-8

Table: Player A's payoff matrix

Solution: We use the maximin-minimax principle to determine the optimal strategy.

Player B

		I	II	III	Row minimum	
	I	-2	15	-2	-2	←Maximin
Player A	II	-5	-6	-4	-6	
	III	-5	20	-8	-8	
	Column maximum	-2	20	-2		
		↑Minimax		↑Minimax		

The game has two saddle points at positions (1, 1) and (1, 3).

I. The best strategy for player A is I.
II. The best strategy for player B is either I or III.
III. The value of the game is -2 for player A and +2 for player B.

MIXED STRATEGIES: Game without a Saddle Point

If *maximin value* is not equal to *minimax value* then the game is said to have no saddle point. In such a case, both the players must determine an optimal mixture of strategies to find an equilibrium point. The optimal strategy mixture for each player may be determined by assigning to each strategy its probability of being chosen. The strategies so determined are called **mixed strategies**.

The value of the game obtained by the use of mixed strategies represents the least payoff which player A can expect to win and the least payoff which player B can expect to lose. The expected payoff to a player in a game with payoff matrix $[v_{ij}]_{m\times n}$ can be defined as

$$E(p,q) = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i v_{ij} q_j = pvq^T$$

where $\mathbf{p} = (p_1, p_2, p_3, ..., p_m)$ and $\mathbf{q} = (q_1, q_2, q_3, ..., q_n)$ denote probabilities or relative frequency with which strategy is chosen from the list of strategies associated with m strategies of player A and n strategies of player B, respectively. Obviously, $p_i \geq 0$ (i = 1, 2, ..., m), $q_j \geq 0$ (j = 1, 2, ...,n) and $p_1 + p_2 + ... + p_m = 1$; $q_1 + q_2 + ... + q_n = 1$.

Theorem 1.2: For any 2×2 two-person zero-sum game without any saddle point having the payoff matrix for Player A given.

	B_1	B_2
A_1	v_{11}	v_{12}
A_2	v_{21}	v_{22}

The optimal mixed strategies $S_A = \begin{bmatrix} A1 & A2 \\ p1 & p2 \end{bmatrix}$ and $S_B = \begin{bmatrix} B1 & B2 \\ q1 & q2 \end{bmatrix}$ are determined by $\frac{p_1}{p_2} = \frac{v_{22}-v_{21}}{v_{11}-v_{12}}, \frac{q_1}{q_2} = \frac{v_{22}-v_{12}}{v_{11}-v_{21}}$ where $p_1 + p_2 = 1$ and $q_1 + q_2 = 1$.

The value of the game to A is given by $v = \frac{v_{11}v_{22} - v_{21}v_{12}}{v_{11} + v_{22} - (v_{12} + v_{21})}$.

Proof: Let a mixed strategy for player A be given by $S_A = \begin{bmatrix} A1 & A2 \\ p1 & p2 \end{bmatrix}$, where $p_1 + p_2 = 1$.

Thus, if player B moves B_1 then the net expected gain of A will be $E_1(p) = v_{11}p_1 + v_{21}p_2$ and if B moves B_2, the net expected gain of A will be $E_2(p) = v_{12}p_1 + v_{22}p_2$.

Similarly, if B plays his mixed strategy $S_B = \begin{bmatrix} B1 & B2 \\ q1 & q2 \end{bmatrix}$, where $q_1 + q_2 = 1$, then B's net expected loss will be $E_1(q) = v_{11}q_1 + v_{12}q_2$, if player A plays A_1, and $E_2(q) = v_{21}q_1 + v_{22}q_2$ if A plays A_2. The expected gain of player A, when B chooses his moves with probabilities q_1 and q_2, is given by $E(p,q) = q_1[v_{11}p_1 + v_{21}p_2] + q_2[v_{12}p_1 + v_{22}p_2]$. Player A would always try to mix his moves with such probabilities so as to maximize his expected gain.

Now, $E(p,q) = q_1[v_{11}p_1 + v_{21}(1-p_1)] + (1-q_1)[v_{12}p_1 + v_{22}(1-p_1)]$

$= [v_{11} + v_{22} - (v_{12} + v_{21})]p_1q_1 + (v_{12} - v_{22})p_1 + (v_{21} - v_{22})q_1 + v_{22}$

$= \lambda\left(p_1 - \frac{v_{22}-v_{21}}{\lambda}\right)\left(q_1 - \frac{v_{22}-v_{12}}{\lambda}\right) + \frac{v_{11}v_{22} - v_{12}v_{21}}{\lambda}$

Where $\lambda = v_{11} + v_{22} - (v_{12} + v_{21})$.

We see that if A chooses $p_1 = \frac{v_{22}-v_{21}}{\lambda}$, he ensures an expected gain of at least $(v_{11}v_{22} - v_{12}v_{21})/\lambda$. Similarly, if B chooses $q_1 = \frac{v_{22}-v_{12}}{\lambda}$, then he can limit his expected loss to at most $(v_{11}v_{22} - v_{12}v_{21})/\lambda$. These choices of p_1 and q_1 will thus be optimal to the two players. Thus, we get

$p_1 = \frac{v_{22}-v_{21}}{\lambda} = \frac{v_{22}-v_{21}}{v_{11} + v_{22} - (v_{12}+v_{21})}$ and $p_2 = 1 - p_1 = \frac{v_{11}-v_{12}}{v_{11} + v_{22} - (v_{12}+v_{21})}$

$q_1 = \frac{v_{22}-v_{12}}{\lambda} = \frac{v_{22}-v_{12}}{v_{11} + v_{22} - (v_{12}+v_{21})}$ and $q_2 = 1 - q_1 = \frac{v_{11}-v_{21}}{v_{11} + v_{22} - (v_{12}+v_{21})}$ and

$$v = \frac{v_{11}v_{22} - v_{21}v_{12}}{v_{11} + v_{22} - (v_{12} + v_{21})}$$

Hence we have

$$\frac{p_1}{p_2} = \frac{v_{22}-v_{21}}{v_{11}-v_{12}}, \frac{q_1}{q_2} = \frac{v_{22}-v_{12}}{v_{11}-v_{21}}, v = \frac{v_{11}v_{22} - v_{21}v_{12}}{v_{11} + v_{22} - (v_{12}+v_{21})} \qquad (1.1)$$

Note: The above formula for p_1, p_2, q_1, q_2 and v are valid only for 2×2 games without saddle point.

Example 1.3: Suppose that in a game of matching coins with two players, one player wins Re 2 when there are 2 heads, and gets nothing when there are 2 tails and loses Re 1 when there are one head and one tail. Determine the best strategies for each player and the value of the game.

Solution: The payoff matrix for player A is given in the below table.

		Player B	
		H	T
Player A	H	2	-1
	T	-1	0

Table: Player A's payoff matrix

The game has no saddle point. Let the player A plays H with probability x and T with probability 1-x. Then, if the player B plays H, then A's expected gain is

$$E(A, H) = x(2) + (1-x)(-1) = 3x - 1.$$

If the player B plays T, A's expected gain is

$$E(A,T) = x(-1) + (1-x)0 = -x.$$

If the player A chooses x such that E (A, H) = E(A, T) = E(A) say, then this will determine best strategy for him. Thus we have 3x-1 = -x or x = ¼. Therefore, the best strategy for the player A is to play H and T with probability ¼ and ¾, respectively. Therefore, the expected gain for player A is

$$E(A) = \tfrac{1}{4}(2) + \tfrac{3}{4}(-1) = -\tfrac{1}{4}.$$

The same procedure can be applied for player B. If the probability of B's choice of H is y and that of T is 1-y then for the best strategy of the player B,

$$E(B, H) = E(B, T)$$

which gives y = ¼. Therefore, 1-y = ¾.

Thus, A's optimal strategy is (¼, ¾) and B's optimal strategy is (¼, ¾). The expected value of the game is

-1/4 to the player A.

This result can also be obtained directly using the formulae (1.1).

Principle of Dominance

The principle of dominance states that if the strategy of a player dominates over the other strategy in all conditions then the later strategy is ignored because it will not affect the solution in any way. Determination of superior or inferior strategy is based upon objective of the player. Since each player is to select his best strategy, the inferior strategies can be eliminated. In other words, ineffective rows and columns can be deleted from the game matrix and only effective rows and columns are retained in the reduced matrix.

For deleting the ineffective rows and columns, the following general rules are to be followed:

1. If all the elements of a row (say i^{th} row) of a payoff matrix are less than or equal to the corresponding elements of any other row (say j^{th} row) then the player A will never choose the i^{th} strategy. Therefore, ith row is dominated by j^{th} row. Hence delete i^{th} row.
2. If all the elements of a column (say j^{th} column) of a payoff matrix are greater than or equal to the corresponding elements of any other column (say i^{th} column) then j^{th} column is dominated by i^{th} column. So delete j^{th} column.
3. A pure strategy of a player may also be dominated if it is inferior to some convex combination of two or more pure strategies. As a particular case, if all the elements of a column are greater than or equal to the average of two or more other columns then this column is dominated by the group of columns. Similarly, if all the elements of row are less than or equal to the average of two or more rows then this row is dominated by the group of rows.

Example: Consider the following 4×4 payoff matrix

		Player B			
		I	II	III	IV
Player A	I	3	5	4	2
	II	5	6	2	4
	III	2	1	4	0
	IV	3	3	5	2

Table 2.1: payoff matrix

First, check that the game has no saddle point. Now see that columns I and II are dominated by column IV. So, columns I and II can be deleted. The reduced payoff matrix is given in the table 2.2 which shows that rows I and II are dominated by row IV. So, rows I and III can be deleted and the reduced matrix is shown in the table 2.3 which is a 2×2 payoff matrix. We can now use a suitable method to determine the mixed strategies and the value of the game.

		Player B	
		III	IV
Player A	I	4	2
	II	2	4
	III	4	0
	IV	5	2

Table 2.2

		Player B	
		III	IV
Player A	II	2	4
	IV	5	2

Table 2.3

Solution methods for Game without a saddle point:

1. Algebraic method:

Let v be the value of the game given by the payoff matrix $[v_{ij}]_{m\times n}$ and p_i (i=1, 2, ...,m), q_j (j=1, 2, ..., n) be the probabilities for the optimal mixed strategies of the player A and B respectively which are shown in the table 2.4.

	Playe B				
Player A	B_1	B_2	...	B_n	Probability
A_1	v_{11}	v_{12}	...	v_{1n}	p_1
A_2	v_{21}	v_{22}	...	v_{2n}	p_2
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
A_m	v_{m1}	v_{m2}	...	v_{mn}	p_m
Probability	q_1	q_2	...	q_n	

Table 2.4

The expected gain of A when he uses his i[th] strategy and B uses his j[th] strategy is $\sum_{i=1}^{m} a_{ij}x_i$. As A expects at least v, the value of the game, we should have

$$\sum_{i=1}^{m} v_{ij}p_i \geq v, \text{ j=1, 2, ..., n where } \sum_{i=1}^{m} p_i = 1 \text{ and } p_i \geq 0 \text{ for all i.} \quad (2.1)$$

Similarly, the expected loss to player B, we should have for his choice

$$\sum_{j=1}^{n} v_{ij}q_j \leq v, \text{ i=1, 2, ..., m where } \sum_{j=1}^{n} q_j = 1 \text{ and } q_j \geq 0 \text{ for all j.} \quad (2.2)$$

To get the values of p_i's and q_j's, the above inequalities are considered as equations and are then solved for given unknowns. However, if the system of equations are found to be not consistent, this implies that one of the inequalities is a strictly inequality. Then we apply trial and error method to find the solution of a set of equations with a few strict inequalities as explained in the following examples although the method is quite lengthy.

Here we state two theorems (without proof) which are the direct implications of the duality theory and which will make the computation easier.

Theorem 2.1: *If for some $j = k$ there be a strict inequality, say $v_{1k}p_1 + v_{2k}p_2 + \ldots + v_{mk}p = > v$, then the corresponding $q_k = 0$; similarly, if for some $i = r$, there be a strict Inequality, say, $v_{r1}q_1 + v_{r2}q_2 + \ldots. + v_{rn}q = < v$, then the corresponding $p_r = 0$.*

Theorem 2.2: *If one player's optimal strategy consists of exactly r strategies with non-zero probabilities, then the optimal strategy of the other player also will involve r pure strategies.*

Example 2.1; Find the value and optimal strategies for two players of the rectangular game whose payoff matrix is given in the following table.

	Player B			
Player A	B_1	B_2	B_3	Probability
A_1	1	-1	-1	p_1
A_2	-1	-1	3	p_2
A_3	-1	2	-1	p_3
Probability	q_1	q_2	q_3	

Solution: First, it is seen that game does not have a saddle point. Also, this game cannot be reduced to (2×2) by the property of dominance. Hence, this game can be solved by the algebraic method.

Let (p_1, p_2, p_3) and (q_1, q_2, q_3) denote the optimal probabilities for mixed strategies for players A and B, respectively, and v be the value of the game. Now from the given payoff matrix we get the following relationship for player A and B respectively.

For player A	For player B
$p_1 - p_2 - p_3 \geq v$	$q_1 - q_2 - q_3 \leq v$
$-p_1 - p_2 + 2p_3 \geq v$	$-q_1 - q_2 + 3q_3 \leq v$
$-p_1 + 3p_2 - p_3 \geq v$	$-q_1 + 2q_2 - q_3 \leq v$

Also the additional restrictions for the probabilities are $p_1 + p_2 + p_3 = 1$; $q_1 + q_2 + q_3 = 1$ and $p_i, q_i \geq 0$ where $i = 1, 2, 3$.

Suppose all inequalities hold as equations, then we get

For player A

$p_1 - p_2 - p_3 = v$
$-p_1 - p_2 + 2p_3 = v$
$-p_1 + 3p_2 - p_3 = v$
$p_1 + p_2 + p_3 = 1$

For player B

$q_1 - q_2 - q_3 = v$
$-q_1 - q_2 + 3q_3 = v$
$-q_1 + 2q_2 - q_3 = v$
$q_1 + q_2 + q_3 = 1$

Solving these equations we get,

$p_1 = 6/13$, $p_2 = 3/13$, $p_3 = 4/13$;

$q_1 = 6/13$, $q_2 = 4/13$, $q_3 = 3/13$

and $v = -1/13$.

Hence, the solution of the game is:

Optimal mixed strategy for the player A is (6/13, 3/13, 4/13).

Optimal mixed strategy for the player B is (6/13, 4/13, 3/13).

The value of the game to the player A is = -1/13.

2. Arithmetic method

This method is also known as odds method. The steps of this method are as follows:

1. Find the difference of the values in cells (1, 1) and (1, 2) of the first row and place it against the second row of the matrix.
2. Find the difference of the values in cells (2, 1) and (2, 2) of the second row and place it against the first row of the matrix.
3. Find the difference of the values in cell (1, 1) and (2, 1) of the first column and place it below the second column of the matrix.
4. Similarly, find the difference of the values in cells (1, 2) and (2, 2) of the second column and place it below the first column.

The above odds or differences are taken as positive (ignoring the negative sign) and are put in the payoff matrix as shown in the below table.

		Player B		
		B_1	B_2	Odds
Player A	A_1	v_{11}	v_{12}	$v_{21} - v_{22}$
	A_2	v_{21}	v_{22}	$v_{11} - v_{12}$
	Odds	$v_{12} - v_{22}$	$v_{11} - v_{21}$	

Then the probabilities for the moves A1, A2, B1 and B2 are calculated as

A1 = $\frac{v_{21} - v_{22}}{(v_{21} - v_{22}) + (v_{11} - v_{12})}$, A2 = $\frac{v_{11} - v_{12}}{(v_{21} - v_{22}) + (v_{11} - v_{12})}$, B1 = $\frac{v_{12} - v_{22}}{(v_{12} - v_{22}) + (v_{11} - v_{21})}$, B2 = $\frac{v_{11} - v_{21}}{(v_{21} - v_{22}) + (v_{11} - v_{12})}$.

The value of the game is determined as $v = \frac{v_{11}(v_{21} - v_{22}) + v_{21}(v_{11} - v_{12})}{(v_{21} - v_{22}) + (v_{11} - v_{12})}$.

3. Graphical method for solving (2×n) games

This method is applicable to only those games in which the maximizing player has two strategies only. The following are the steps involved in this method for solving a 2×n game:

1. Let the probabilities of the two alternatives of the player A be p_1 and $(1-p_1)$. Then A's expected payoff for each of the pure strategies of B is $E_j(p) = a_{1j}p_1 + a_{2j}p_2 = (a_{1j} - a_{2j})p_1 + a_{2j}$, $j = 1, 2, \ldots, n$; a_{ij}'s are the elements of A's payoff matrix, $i = 1, 2$; $j = 1, 2, \ldots, n$.
2. Draw the line segments for the expected payoffs between two vertical lines unit distance apart. These line segments represent A's expected gain due to B's pure move.
3. For player A, the objective is to maximize the minimum expected gain. The highest point of the intersection of the gain lines in the 'lower envelop' represents the maximin value of the game for Player A.
4. The two strategies of player B corresponding to those lines which pass through the maximin point helps in reducing the size of the game to (2×2).

In case of minimizing player B, the point where maximum loss is minimized is justified. This will be **the lowest point** at the intersection of the lines in the '**upper envelop**'.

Example 2.2: Solve the game graphically whose payoff matrix for the player A is given below.

		Player B				
		B1	B2	B3	B4	B5
Player A	A1	-5	5	0	-1	8
	A2	8	-4	-1	6	-5

Solution: This game does not have a saddle point. Let p_1 be the probability of player A selecting strategy A_1 and hence $p_2 = (1-p_1)$ be the probability of A selecting strategy A_2. The player A's expected payoffs corresponding to the player B's pure strategies are given in the below table.

B's pure strategy	A's expected payoff $E(p_1)$
B_1	$E_1(p_1) = -5(p_1) + 8(1-p_1) = -13p_1 + 8$
B_2	$E_2(p_1) = 5(p_1) - 4(1-p_1) = 9p_1 - 4$
B_3	$E_3(p_1) = 0(p_1) - 1(1-p_1) = p_1 - 1$
B_4	$E_4(p_1) = -1(p_1) + 6(1-p_1) = -7p_1 + 6$
B_5	$E_5(p_1) = 8(p_1) - 5(1-p_1) = 13p_1 - 5$

These five expected payoff lines can be plotted on a graph.

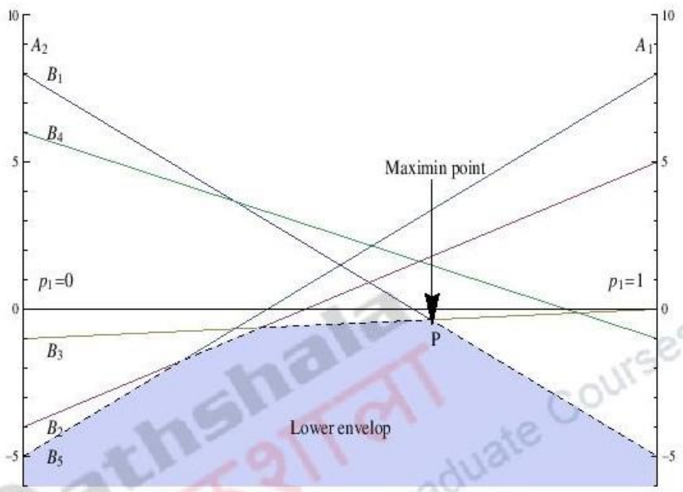

Fig. 2.1: Graphical solution of a $(2 \times n)$ game

First, draw two parallel lines one unit apart and mark a scale on each. These two lines represents two strategies available to the player A. Draw lines to represent each of player B's strategies. For example, to represent B_1 strategy, join mark -5 on A_1 to mark 8 on scale A_2; to represent B_2 strategy, join mark 5 on A_1 to mark -4 on scale A_2.

Since the expected payoff E (p_1) is the function of p_1 alone, these five expected payoff lines can be drawn by taking p_1 as x-axis and E (p_1) as y-axis. The highest point (P) on the lower shaded envelop shown in the above graph gives the maximum expected payoff among the minimum expected payoffs of player A. Since the strategies B_1 and B_3 intersect at the maximin point P, these strategies are selected and the resultant 2×2 matrix is obtained as shown in the below table.

		Player B	
		B_1	B_2
Player A	A_1	-5	0
	A_2	8	-1

The optimum payoff to player A can now be obtained by setting E_1 and E_3 equal and solving for p_1, i.e., $-13p_1 + 8 = p_1 - 1$ or $p_1 = 9/14$ and $1 - p_1 = 5/14$. Substituting these values in E_1 and E_3 we have the value of the game v = -5/14. The optimal strategy mix of player B can also be found in the same manner as for player A. If the probabilities of B's selecting B_1 and B_3 are denoted by q_1 and q_3 then solving the equations $-5q_1 + 0q_3 = v$, $8q_1 - q_3 = v$ and $q_1 + q_3 = 1$ we get $q_1 = 1/14$ and $q_3 = 13/14$. Therefore, the solution of the game is obtained as follows:

(i) The player A's optimal mixed strategy is (9/14, 5/14).

(ii) The player B's optimal mixed strategy is (1/14, 0, 13/14, 0, 0).
(iii) The value of the game to the player A is v = -5/14.

4. Graphical method for solving (m×2) games:

This method is applicable to only those games in which the minimizing player has two strategies only. Consider the following example:

Example 3.1: Solve the game graphically whose payoff matrix for the player A is given by

Player B

		I	II
Player A	I	2	4
	II	2	3
	III	3	2
	IV	-2	6

Solution: The game does not have a saddle point. Let q_1 and q_2 (= $1 - q_1$) be the mixed strategies of the player B. The expected payoffs for the player B are shown in the below table.

A's pure strategy	B's expected payoff $E(p_1)$
A_1	$E_1(q_1) = 2(q_1) + 4(1-q_1) = -2q_1 + 4$
A_2	$E_2(q_1) = 2(q_1) + 3(1-q_1) = -q_1 + 3$
A_3	$E_3(q_1) = 3(q_1) + 2(1-q_1) = q_1 + 2$
A_4	$E_4(q_1) = -2(q_1) + 6(1-q_1) = -8q_1 + 6$

Player B

		B_1	B_2
Player A	A_1	2	4
	A_3	3	2

Now, plot these expected payoff lines as shown in the below figure. In this case, the minimax point is determined as **the lowest point** P of the **upper envelop**. Lines intersecting at the minimax point P correspond to player A's pure strategies A_1 and A_3. This indicates that $p_2 = p_4 = 0$. The reduced game is given below.

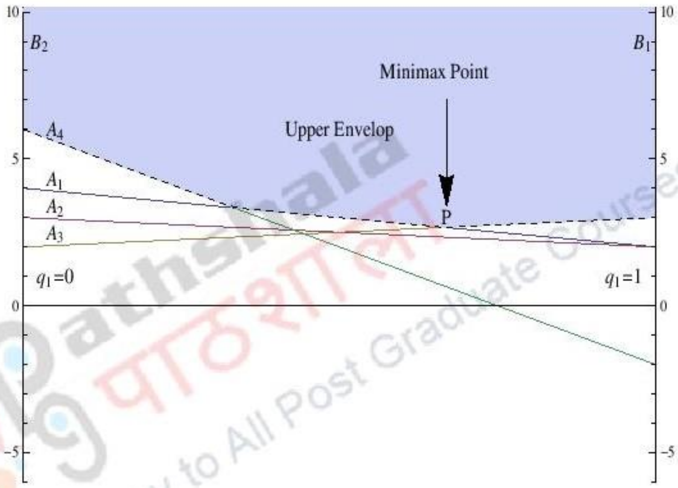

Fig. 3.1: Graphical solution for a $(m \times 2)$ game

Now, to solve this (2×2) game, we solve the following simultaneous equations:

$$2p_1 + 3p_3 = v,\ 4p_1 + 2p_3 = v,\ p_1 + p_3 = 1 \text{ (for player A)}$$

$$2q_1 + 4q_2 = v,\ 3q_1 + 2q_2 = v,\ q_1 + q_2 = 1 \text{ (for player B)}$$

The solution of the game thus obtained is as follows:

(i) The player A's optimal mixed strategy is (1/3, 0, 2/3, 0).
(ii) The player B's optimal mixed strategy is (2/3, 1/3).
(iii) The value of the game to the player A is v = 8/3.

5. Linear programming method (using simplex method):

Every finite two-person zero-sum game can be expressed as a linear programme and conversely, every linear programme can be represented as a two-person zero-sum game. Consider a game problem with the payoff matrix $(v_{ij})_{m\times n}$. The maximizing player A has m mixed strategies which he chooses with probabilities p_1, p_2, ..., p_m, respectively. The expected gain to A when the minimizing player B chooses his j^{th} course of action out of his n courses is $\sum_{i=1}^{m} v_{ij}p_i$. If $\underbrace{min}_{j}(\sum_{i=1}^{m} v_{ij}p_i) = v$, then A expects to get at least v. A will choose his strategies with such probabilities as he maximizes this least guaranteed gain v. Now

$$\sum_{i=1}^{m} v_{ij}p_i \geq v, for\ all\ j\ and\ \sum_{i=1}^{m} p_{i=1} \qquad (I)$$

as v is the minimum of all expected gains and the value of the game is clearly the maximum value v, if it exists. Thus p_1, p_2, ..., p_m are to be determined so as to satisfy the equality constraint and maximize v.

To put this problem in standard LP problem, we divide (I) by v, which is positive. If v is not positive, which will be indicated by the presence of some negative elements in the payoff matrix, and then we add to all the elements of the payoff matrix a sufficiently large positive quantity c such that v becomes positive. This operation does not change the optimal solution but only increases the value of the game by c.

Now, dividing (I) by v (>0) and setting $\frac{p_i}{v} = P_i$ (i = 1, 2, ..., m), we write the problem as

$$v_{1j}P_1 + v_{2j}P_2 + \ldots + v_{mj}P_m \geq 1, \quad j = 1, 2, \ldots, n$$

$$\text{and} \quad P_1 + P_2 + \ldots + P_m = \frac{1}{v}$$

$$\text{with} \quad P_1, P_2, \ldots, P_m \geq 0$$

Now, the maximization of v is equivalent to the minimization of $\frac{1}{v}$. Hence we can state the problem for A as

$$Minimize \frac{1}{v} = P_1 + P_2 + \ldots + P_m \qquad (II)$$

$$v_{1j}P_1 + v_{2j}P_2 + \ldots + v_{mj}P_m \geq 1, j = 1, 2, \ldots, n$$

$$with \; P_1\, P_2, \ldots\ldots, P_m \; \geq 0$$

This is an LPP (Linear programming problem) written from the point of view of A.

Considering the problem from the point of view of B who will expect to minimize all the expected losses for him, we arrive in a similar manner at the following LPP:

$$\text{Maximize} \frac{1}{v} = Q_1 + Q_2 + \cdots + Q_n \qquad (III)$$

$$v_{i1}Q_1 + v_{i2}Q_2 + \cdots + v_{in}Q_n \leq 1, i = 1, 2, \ldots, m$$

$$\text{With } Q_j = \frac{q_j}{v} \geq 0, j = 1, 2, \ldots, n$$

where q_j's are mixed strategies of B. Having found P_i's and Q_j's, we can find p_i's and q_j's from the relations $p_i = vP_i$ and $q_j = vQ_j$. Finally the value of the original game is obtained by subtracting c, if needed.

It is seen that the problems (II) and (III) as stated above are dual to one another. Thus the optimal solution of one problem will afford an optimal solution for the other. Player B's problem a can be solved by regular simplex method while that of A can be solved by the method of duality.

Example: Use linear programming technique to determine the best strategies for both the players involved in a game whose payoff matrix is given below:

		Player B		
		B_1	B_2	B_3
Player A	A_1	3	-4	2
	A_2	1	-7	-3
	A_3	-2	4	7

Solution: First find the minimax ($\overline{v}$) and maximin ($\underline{v}$) values of the game. The minimax value = -2 and maximin value = 3. Thus the game has no saddle point. The value of the game lies between -2 and +3. It is possible that the value of game may be negative or zero. So, a constant k = 3 is added to all the elements of the payoff matrix. The modified payoff matrix given in the below table (II). Let v be the value of the game; p_1, p_2 and p_3 be the probabilities of selecting strategies A_1, A_2 and A_3, respectively; and q_1, q_2 and q_3 be the probabilities of selecting strategies B_1, B_2 and B_3 respectively.

Player B

	B_1	B_2	B_3	Minimum
Player A A_1	3	-4	2	-4
A_2	1	-7	-3	-7
A_3	-2	4	7	-2
Maximum	3	4	7	

Table (I)

Player B

	B_1	B_2	B_3	Probability
Player A A_1	6	-1	5	p_1
A_2	4	-4	0	p_2
A_3	1	7	10	p_3
Probability	q_1	q_2	q_3	

Table (II)

Player A's objective is to maximize the expected gains, which can be achieved by maximizing v, i.e., it might gain more than v if player B adopts a poor strategy. Hence, the expected gains for the player A will be as follows:

$$6p_1 + 4p_2 + p_3 \geq v$$

$$-p_1 - 4p_2 + 7p_3 \geq v$$

$$5p_1 + 0p_2 + 10p_3 \geq v$$

$$p_1 + p_2 + p_3 = 1$$

$$p_1, p_2, p_3 \geq 0$$

Dividing the above constraints by v, we get

$$6\frac{p_1}{v} + 4\frac{p_2}{v} + \frac{p_3}{v} \geq 1$$

$$-\frac{p_1}{v} - 4\frac{p_2}{v} + 7\frac{p_3}{v} \geq 1$$

$$5\frac{p_1}{v} + 0\frac{p_2}{v} + 10\frac{p_3}{v} \geq 1$$

$$\frac{p_1}{v} + \frac{p_2}{v} + \frac{p_3}{v} = \frac{1}{v}$$

To simplify the problem, we put $\frac{p_1}{v} = x_1, \frac{p_2}{v} = x_2$ and $\frac{p_3}{v} = x_3$. In order to maximize v, player A can

Minimize $\frac{1}{v} = x_1 + x_2 + x_3$

subject to $6x_1 + 4x_2 + x_3 \geq 1$

$$-x_1 - 4x_2 + 7x_3 \geq 1$$

$$5x_1 + 0x_2 + 10x_3 \geq 1$$

$$x_1, x_2, x_3 \geq 0$$

Player B's objective is to minimize its expected losses, which can be reduced by minimizing v. Player B might lose less than v if player A adopts a poor strategy. Hence, the expected loss for player B will be as follows:

$$6q_1 - q_2 + 5q_3 \leq v$$

$$4q_1 - 4q_2 + 0q_3 \leq v$$

$$q_1 + 7q_2 + 10q_3 \leq v$$

$$q_1 + q_2 + q_3 = 1$$

$$\text{and } q_1, q_2, q_3 \geq 0$$

Dividing the above constraints by v, we get

$$6\frac{q_1}{v} - \frac{q_2}{v} + 5\frac{q_3}{v} \leq 1$$

$$4\frac{q_1}{v} - 4\frac{q_2}{v} + 0\frac{q_3}{v} \leq 1$$

$$\frac{q_1}{v} + 7\frac{q_2}{v} + 10\frac{q_3}{v} \leq 1$$

$$\frac{q_1}{v} + \frac{q_2}{v} + \frac{q_3}{v} = \frac{1}{v}$$

To simplify the problem, we put $\frac{q_1}{v} = y_1, \frac{q_2}{v} = y_2$ and, $\frac{q_3}{v} = y_3$. In order to minimize v, player B can

$$\text{Maximize } \frac{1}{v} = y_1 + y_2 + y_3$$

$$\text{subject to } 6y_1 - y_2 + 5y_3 \leq 1$$

$$4y_1 - 4y_2 + 0y_3 \leq 1$$

$$y_1 + 7y_2 + 10y_3 \leq 1$$

$$y_1, y_2, y_3 \geq 0$$

We introduce slack variables to convert inequalities to equalities. The problem then becomes

$$\text{Maximize } \frac{1}{v} = y_1 + y_2 + y_3 + 0y_4 + 0y_5 + 0y_6$$

$$\text{subject to } 6y_1 - y_2 + 5y_3 + y_4 = 1$$

$$4y_1 - 4y_2 + 0y_3 + y_5 = 1$$

$$y_1 + 7y_2 + 10y_3 + y_6 = 1$$

$$y_1, y_2, y_3, y_4, y_5, y_6 \geq 0$$

			$C_j \rightarrow$	1	1	1	0	0	0	Minimum
C_B	B	Y_B	b	a_1	a_2	a_3	a_4	a_5	a_6	Ratio
0	a_4	y_4	1	6	-1	5	1	0	0	1/6
0	a_5	y_5	1	4	-4	0	0	1	0	1/4
0	a_6	y_6	1	1	7	10	0	0	1	1
	Z_j - C_j			-1	-1	-1	0	0	0	
1	a_1	y_1	1/6	1	- 1/6	5/6	1/6	0	0	
0	a_5	y_5	1/3	0	-10/3	-10/3	- 2/3	1	0	
0	a_6	y_6	5/6	0	43/6	55/6	- 1/6	0	1	5/43
	Z_j - C_j			0	-7/6	- 1/6	1/6	0	0	
1	a_1	y_1	8/43	1	0	45/43	7/43	0	1/43	
0	a_5	y_5	31/43	0	0	40/43	-32/43	1	20/43	
1	a_2	y_2	5/43	0	1	55/43	-1/43	0	6/43	
	Zj-Cj			0	0	57/43	6/43	0	7/43	

Simplex Table

Now, using simplex method, we obtain the optimal results as (see the above simplex table)

$y_1 = \frac{8}{43}, y_2 = \frac{5}{43}, y_3 = 0 \; and \; v = \frac{43}{13}$. Then we have

$$q_1 = v \times y_1 = \frac{43}{13} \times \frac{8}{43} = \frac{8}{13}$$

$$q_2 = v \times y_2 = \frac{43}{13} \times \frac{5}{43} = \frac{5}{13}$$

$$q_3 = v \times y_3 = \frac{43}{13} \times 0 = 00$$

Hence, player B's optimal strategy is $(\frac{8}{13}, \frac{5}{13}, 0)$.

From the simplex table, we also find that $x_1 = \frac{6}{43}, x_2 = 0 \; and \; x_3 = 7/43$. Then we have

$$p_1 = v \times x_1 = \frac{43}{13} \times \frac{6}{43} = \frac{6}{13}$$

$$p_2 = v \times x_2 = \frac{43}{13} \times 0 = 0$$

$$p_3 = v \times x_3 = \frac{43}{13} \times \frac{7}{43} = \frac{7}{13}$$

Hence, player A's optimal strategy is $(\frac{6}{13}, 0, \frac{7}{13})$.